Inflation of a Viscoelastic Ellipsoidal Neo-Hookean Membrane

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This paper concerns the inflation of an ellipsoidal viscoelastic nonlinear membrane. The creep and relaxation phenomena are obtained for constant inflating pressures and constant deformations. The governing equations are written in terms of three new variables and are reduced to three partial differential integral equations. These equations are further reduced to three first-order ordinary equations with explicit derivatives by employing the trapezoidal rule for integrations. The constitutive equations developed by Pipkin and Rogers[2] are used in this paper.

Introduction

NONLINEAR membrane problems are found in a wide variety of engineering applications. Yang and Feng[1] demonstrated that with the aid of modern digital computers, all axisymmetric nonlinear elastic membrane problems can be solved numerically. These membranes usually consist of rubber-like materials which exhibit nonlinear viscoelastic properties in large deformation applications. However, due to the inherent complexities associated with nonlinearity in geometry and in material, very few nonlinear viscoelastic membrane problems have been investigated.

This paper concerns a nonlinear viscoelastic ellipsoidal membrane inflated by internal pressure. It is assumed that the ellipsoidal membrane is composed of homogeneous isotropic incompressible viscoelastic material, that the initial configuration of the membrane is axisymmetric, and that the load is applied axisymmetrically so that the deformed configurations of the membrane are also axisymmetric. It is further assumed that the membrane is relatively thin compared with its other dimensions so that bending effects can be neglected. As a result of the preceding assumptions there are no strain variations across the midsurface of the membrane, and the physical quantities pertaining to points not on the deformed midsurface are the same as those on the deformed midsurface.

Based on these assumptions, the governing equations for the variables describing the deformed configurations of an ellipsoidal nonlinear viscoelastic membrane that is initially axisymmetric are reduced to a set of three first-order partial differential integral equations. These governing equations are further reduced to three first-order ordinary differential equations with explicit derivatives by employing the trapezoidal rule for the integrations. These ordinary differential equations correspond to the ordinary differential equations obtained by Yang and Feng for the axisymmetric nonlinear elastic membrane problems and may be integrated by any standard numerical method.

The constitutive equations for the stress and strain relations used for the viscoelastic membrane are those developed by Pipkin and Rogers.[2] Their theory was later simplified to a specific model by Wineman.[3] In this paper the constitutive equations are further simplified for materials that behave like neo-Hookean elastic material at times zero and infinity.

Results for both creep and relaxation phenomena are obtained. The creep phenomenon of the viscoelastic membrane is observed by keeping the pressure constant. The relaxation phenomenon of the membrane is observed by keeping the deformed configuration of the membrane unchanged. The deformed configurations of the membrane for creep, and the variation of the inflating pressure vs time for relaxation are presented.

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Deformation Analysis

Quarters of the midsurfaces of undeformed and deformed meridian sections of the viscoelastic ellipsoidal membrane are shown in Fig. 1. The coordinates that define the configurations of a membrane before deformation are (r, ψ, s). The coordinates that define the configurations of a membrane after deformation at time t are (x, ψ, y). For axisymmetric deformation, ψ remains unchanged and the axes of symmetry, y and s, coincide during deformation. The coordinates r and s, for an undeformed meridian midsurface of an ellipsoidal membrane, are written in the following in terms of the angle ϕ, measured clockwise from the s axis as shown in Fig. 1 ($90° - \phi$ is the latitude angle)

$$r = a\sin\phi \tag{1a}$$

$$s = ae\cos\phi \tag{1b}$$

where $e = b/a$ is the aspect ratio. The letters a and b are the semimajor and the semiminor axis of the ellipse, respectively.

A pressure is applied to the membrane at time $t = 0^+$. A point $p(\mathrm{r}, \psi, s)$ on the undeformed membrane is deformed to $p^D(x, \psi, y, t)$ on the deformed membrane. The deformation is continuous and the mapping is in one-to-one correspondence; hence

$$x = x(\phi, t) \tag{2a}$$

$$y = y(\phi, t) \tag{2b}$$

Two adjacent elements intersecting at point p on the meridian and circumferential circles of the undeformed membrane have arc lengths $\partial\ell_m = [(\partial r)^2 + (\partial s)^2]^{1/2}$ and $\partial\ell_c = r\partial\psi$, respectively. After deformations, these elements intersect at point p^D. The corresponding arc lengths are $\partial L_m = [(\partial x)^2 + (\partial y)^2]^{1/2}|_{t=t}$ and $\partial L_c = x\partial\psi|_{t=t}$; therefore the principal stretch ratios in the meridian and circumferential directions are, respectively,

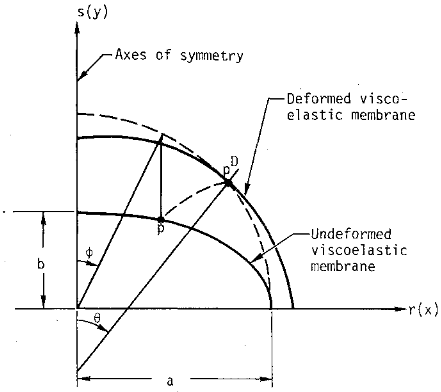

Fig. 1 Geometry of deformations.

$$\lambda_1 = \frac{[x'^2 + y'^2]^{1/2}}{a[\cos^2\phi + e^2\sin^2\phi]^{1/2}} \tag{3a}$$

$$\lambda_2 = \frac{x}{a\sin\phi} \tag{3b}$$

The prime in Eqs. (3) and subsequent equations denotes the partial differentiation with respect to ϕ.

The third principal stretch ratio λ_3 normal to the deformed midsurface at point p^D, is

$$\lambda_3 = \frac{h^D}{h} \tag{4}$$

where h is the undeformed thickness of the membrane at p and h^D is the deformed thickness of the membrane at p^D. For axisymmetric problems, h is a function of ϕ only. For incompressible materials, h is related to λ_1 and λ_2 by the incompressibility condition

$$\lambda_1\lambda_2\lambda_3 = 1 \tag{5}$$

Equations (3) may be solved for y', yielding

$$y' = -[a^2\lambda_1^2(\cos^2\phi + e^2\sin^2\phi) - x'^2]^{1/2} \tag{6}$$

A negative sign is chosen, since y decreases as ϕ increases.

The principal curvatures for the arc lengths on the meridian and circumferential circles of the deformed membrane at time t are, respectively,

$$k_1 = \frac{\partial\theta}{\partial L_m}\bigg|_{t=t}, \quad \text{and} \quad k_2 = \frac{\sin\theta}{x}\bigg|_{t=t} \tag{7}$$

where θ is the angle between the vector normal to the midsurface of a deformed membrane, and the axis of symmetry. Hence

$$\cos\theta = \frac{\partial x}{\partial L_m}\bigg|_{t=t}, \quad \sin\theta = -\frac{\partial y}{\partial L_m}\bigg|_{t=t} \tag{8}$$

Substituting Eqs. (3), (6), and (8) into Eqs. (7), the principal curvatures at $t=t$ may be written in terms of the two variables λ_1 and x and the independent variables ϕ and t, as follows

$$k_1 = \frac{-\lambda_1 x'' - x'(\cos^2\phi - e^2\sin^2 c)^{-1}[\lambda_1'(\cos^2\phi + e^2\sin^2\phi) + \lambda_1(e^2-1)\cos\phi\sin\phi]}{a\{a^2\lambda_1^2(\cos^2\phi + e^2\sin^2\phi) - x'^2\}^{1/2}\lambda_1^2(\cos^2\phi + e^2\sin^2\phi)^{1/2}} \tag{9a}$$

$$k_2 = \frac{[a^2\lambda_1^2(\cos^2\phi + e^2\sin^2\phi) - x'^2]^{1/2}}{a\lambda_1 x(\cos^2\phi + e^2\sin^2\phi)^{1/2}} \tag{9b}$$

Constitutive Equations

Constitutive equations for finite deformation of a viscoelastic material that were developed by Pipkin and Rogers and used by Wineman are simplified in this paper so that the material exhibits neo-Hookean elastic material properties when $t=0^+$ and $t\to\infty$. The stresses in the meridian and circumferential directions are, respectively,

$$\sigma_1 = c_0\left(\lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left\{\int_0^t \frac{\partial}{\partial(t-\tau)} c[I(\tau), t-\tau]\mathrm{d}\tau + 1\right\} \tag{10a}$$

$$\sigma_2 = c_0\left(\lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left\{\int_0^t \frac{\partial}{\partial(t-\tau)} c[I(\tau), t-\tau]\mathrm{d}\tau + 1\right\} \tag{10b}$$

where c_0 is a material constant in the units of stress and $I(\tau)$ is the first strain invariant, defined by

$$I(\tau) = \lambda_1^2(\tau) + \lambda_2^2(\tau) + 1\lambda_1^2(\tau)\lambda_2^2(\tau) \tag{11}$$

A special form for the integrand is

$$\frac{\partial}{\partial(t-\tau)} c[I(\tau), t-\tau] = -(1-\gamma)\frac{1}{\tau_R} \times[1+\beta(I(\tau)-3)]\exp\left\{-\frac{1}{\tau_R}[1+\beta(I(\tau)-3)](t-\tau)\right\} \tag{12}$$

where γ and β are two other dimensionless material constants and τ_R is a relaxation time constant.

In the preceding equations the physical quantities without parentheses denote those evaluated at current time t, and the physical quantities with τ enclosed in parentheses denotes those evaluated at time $\tau(0<\tau\leq t)$. In terms of the variables, λ_1, x, and the independent variable ϕ, Eqs. (10) reduce to

$$\sigma_1 = c_0\left(\lambda_1^2 - \frac{a^2\sin^2\phi}{\lambda_1^2 x^2}\right)\left\{1 - \frac{1-\gamma}{\tau_R}\int_0^t A(\tau)\exp\left[-A(\tau)\frac{t-\tau}{\tau_R}\right]\mathrm{d}\tau\right\} \tag{13a}$$

$$\sigma_2 = c_0\left(\frac{x^2}{a^2\sin^2\phi} - \frac{a^2\sin^2\phi}{\lambda_1^2 x^2}\right)\left\{1 - \frac{1-\gamma}{\tau_R}\int_0^t A(\tau) \times\exp\left[-A(\tau)\frac{t-\tau}{\tau_R}\right]\mathrm{d}\tau\right\} \tag{13b}$$

where

$$A(\tau) = 1 + \beta\left(\lambda_1^2(\tau) + \frac{x^2(\tau)}{a^2\sin^2\phi} + \frac{a^2\sin^2\phi}{\lambda_1^2(\tau)x_2^2(\tau)} - 3\right) \tag{14}$$

It is noteworthy that when $t=0$, Eqs. (13) reduce to

$$\sigma_1 = c_0\left(\lambda_1^2 - \frac{a^2\sin^2\phi}{\lambda_1^2 x^2}\right), \quad \sigma_2 = c_0\left(\frac{x^2}{a^2\sin^2\phi} - \frac{a^2\sin^2\phi}{\lambda_1^2 x^2}\right) \tag{15}$$

and when $t\to\infty$, Eqs. (13) reduce to

$$\sigma_1 = \gamma c_0\left(\lambda_1^2 - \frac{a^2\sin^2\phi}{\lambda_1^2 x^2}\right), \sigma_2 = \gamma c_0\left(\frac{x^2}{a^2\sin^2\phi} - \frac{a^2\sin^2\phi}{\lambda_1^2 x^2}\right) \tag{16}$$

Hence, the viscoelastic material behaves like a neo-Hookean elastic material for times of both zero and infinity. The value γ represents the ratio of the long term modulus to the initial modulus, therefore, when time elapses, the material becomes stiffer for $\gamma > 1$ and softer for $\gamma < 1$.

Governing Equations and Boundary Conditions

The equations of equilibrium in the meridian and normal directions to the midsurface of the deformed membrane are

$$\frac{\partial s_1}{\partial x} + \frac{1}{x}(s_1 - s_2) = 0 \tag{17a}$$

$$k_1 s_1 + k_2 s_2 = p_n \tag{17b}$$

where p_n is the inflating pressure that is applied to the membrane at $t\geq 0^+$. The symbols s_1 and s_2 represent the principal stress resultants in the meridian and circumferential directions, respectively. Their units are force per unit deformed arc length. The principal stress resultants are related to the stresses of the deformed membrane by

$$s_1 = h\lambda_3\sigma_1, \qquad s_2 = h\lambda_3\sigma_2 \tag{18}$$

Equations (17) are written in terms of the deformed coordinates. They must be converted into equations with an undeformed coordinate as the independent variable. Substituting Eqs. (3), (9), (13), and (18), and introducing a new variable W and the following nondimensional quantities,

$$S_1 = \frac{s_1}{c_0 h}, \qquad S_2 = \frac{s_2}{c_0 h} \tag{19a}$$

$$T = \frac{t}{\tau_R}, \qquad \Upsilon = \frac{\tau}{\tau_R} \tag{19b}$$

$$X = \frac{x}{a}, \qquad Y = \frac{y}{a} \tag{19c}$$

$$P = \frac{p_n a}{c_0 h} \tag{19d}$$

Equation (17) reduce to

$$\lambda_1' = \left[\left(\frac{3\sin^2\phi}{\lambda_1^3 x^2} - \lambda_1\right)\frac{\cos\phi}{X} + F_2 W + F_3 + F_4 + F_5\right]\Big/ F_1 \tag{20a}$$

$$X' = W \tag{20b}$$

$$W' = \frac{W[\lambda_1'(\cos^2\phi + e^2\sin^2\phi) + \lambda_1^2(e^2-1)\sin\phi\cos\phi]}{\lambda_1(\cos^2\phi + e^2\sin^2\phi)} + \frac{\lambda_1[\lambda_1^2(\cos^2\phi+e^2\sin^2\phi) - W^2]^{1/2}(\cos^2\phi+e^2\sin^2\phi)^{1/2}}{(\lambda_1^2 - \sin^2\phi/\lambda_1^2 X^2)[1-(1-\gamma)H]}$$

$$\times\left\{\frac{[\lambda_1^2(\cos^2\phi+e^2\sin^2\phi) - W^2]^{1/2}}{\lambda_1 X(\cos^2\phi+e^2\sin^2\phi)^{1/2}}\left(\frac{x^2}{\sin^2\phi} - \frac{\sin^2\phi}{\lambda_1^2 X^2}\right)[1-(1-\gamma)H] - \frac{P\lambda_1 X}{\sin\phi}\right\} \tag{20c}$$

where

$$F_1 = \frac{\sin\phi}{X} + \frac{3\sin^3\phi}{\lambda_1^4 X^3} - (1-\gamma)\beta\left(\lambda_1 - \frac{\sin^2\phi}{\lambda_1^3 X^2}\right)\Delta T \tag{21a}$$

$$F_2 = \left(\frac{\lambda_1\sin\phi}{X^2} - \frac{3\sin^3\phi}{\lambda_1^3 X^4}\right) - \left(\frac{\lambda_1\sin\phi}{X^2} - \frac{1}{\lambda_1\sin\phi}\right)[1-(1-\gamma)H] \tag{21b}$$

$$F_3 = 2\beta(1-\gamma)\int_0^{T-\Delta T}\left(\lambda_1(\Upsilon) - \frac{\sin^2\phi}{\lambda_1^3(\Upsilon)X^2(\Upsilon)}\right)[1-(T-\Upsilon)A(\Upsilon)]\exp(-A(\Upsilon)(T-\Upsilon))\lambda_1'(\Upsilon)\,\mathrm{d}\Upsilon \tag{21c}$$

$$F_4 = 2\beta(1-\gamma)\int_0^{T}\left(\frac{X(\Upsilon)}{\sin^2\phi} - \frac{\sin^2\phi}{\lambda_1^2(\Upsilon)X^3(\Upsilon)}\right)[1-(T-\Upsilon)A(\Upsilon)]\exp(-A(\Upsilon)(T-\Upsilon)W(\Upsilon)\,\mathrm{d}\Upsilon \tag{21d}$$

$$F_5 = -2\beta(1-\gamma)\cos\phi\int_0^{T}\left(\frac{X^2(\Upsilon)}{\sin^3\phi} - \frac{\sin\phi}{\lambda_1^2(\Upsilon)x_2^2(\Upsilon)}\right)[1[1-(T-\Upsilon)A(\Upsilon)]\exp(-A(\Upsilon)(T-(\Upsilon))\,\mathrm{d}\Upsilon \tag{21e}$$

$$H = \int_0^T A(\Upsilon)\exp(-A(\Upsilon)(t-\Upsilon))\,\mathrm{d}\Upsilon \tag{21f}$$

The symbol ΔT represents the current time interval used in numerical calculations. The integration limits for F_3 are from 0 to $T-\Delta T$, since its integrand contains λ_1'. The current value for the integration of F_3 is combined in the last term of F_1. The integrals F_4, F_5, and H do not contain the derivatives of λ_1' or W'; hence, they are integrated from 0 to T. The effects of the viscoelastic properties of the membrane on the current stretch ratios are contributed through these integrations. Equations (20) are applicable to $t \geq 0^+$, except that when $t=0^+$, Eqs. (21) are replaced by

$$F_1 = \frac{\sin\phi}{X} + \frac{3\sin^3\phi}{\lambda_1^4 X^3} \tag{22a}$$

$$F_2 = \frac{1}{\lambda_1\sin\phi} - \frac{3\sin^3\phi}{\lambda_1^3 X^4} \tag{22b}$$

$$F_3 = F_4 = F_5 = H = 0 \tag{22c}$$

The boundary conditions are

$$\phi = 0, \qquad X = 0 \tag{23a}$$

$$\phi = 0, \qquad W = \lambda_1 \tag{23b}$$

$$\phi = \frac{\pi}{2}, \qquad W = 0 \tag{23c}$$

The second boundary condition is obtained from Eq. (6), where $\phi = 0$ and $Y' = 0$.

Numerical Calculations and Results

Equations (20) are three first-order ordinary differential equations with explicit derivatives for $t = 0^+$. They can be integrated by any numerical integration method for the instantaneous response of the ellipsoidal membrane under an inflating pressure. The responses of the membrane for $T > 0^+$ are obtained by dividing the total time into n parts with intervals, $T_k - T_{k-1}$ $(k = 1,\ldots,n)$, where $T_0 = 0^+$ and $T_n = t$. The integrals in Eqs. (21) are integrated by the trapezoidal rule; hence

$$\int_0^T G[\lambda, \lambda(\Upsilon), T-\Upsilon]\,\mathrm{d}\Upsilon \approx \frac{1}{2}\sum_{k=1}^{n}\{G[\lambda(T_n), \lambda(T_k), T_n - T_k] + G[s(T_n), \lambda(T_{k-1}), T_n - T_{k-1}]\} \times (T_k - T_{k-1}) \tag{24}$$

$$\int_0^{T-\Delta T} G[\lambda, \lambda(\Upsilon), T-\Upsilon]\,\mathrm{d}\Upsilon \approx \frac{1}{2}\sum_{k=1}^{n}[G[\lambda(T_n),\lambda(T_k),T_n - T_k] + G\{\lambda(T_n), \lambda(T_{k-1}), T_n - T_{k-1}]\}\times(T_k - T_{k-1}) - \frac{1}{2}G[\lambda(T_n), \lambda(T_n), 0](T_n - T_{n-1}) \tag{25}$$

When the integrals in Eqs. (20) and (21) are replaced by the series defined in Eqs. (24) and (25), the governing equations are reduced to three first-order ordinary differential equations for $T > 0^+$.

The boundary-value problem is changed to an initial-value problem by assuming $W = \lambda_1 = \lambda_0$ at $\phi = 0$, where λ_0 is an assumed value. The assumed boundary conditions and the first boundary condition in Eq. (23) provide three boundary conditions at $\phi = 0$, for three first-order ordinary differential equations. The Runge-Kutta method is employed for calculating the physical quantities for the successive points. The third boundary condition is satisfied by adjusting the assumed value λ_0. The creep and the relaxation problems for a viscoelastic ellipsoid membrane are presented in Figs. 2-6 for an aspect ratio of 0.5. The deformations for the creep problem are obtained by keeping the inflation pressure con-

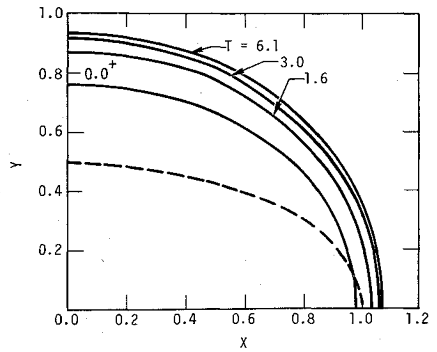

Fig. 2 Profiles of deformed and undeformed ellipsoidal viscoelastic membranes under constant inflating pressure ($p = 1.0$, $\beta = 0.25$, $\gamma = 0.70$).

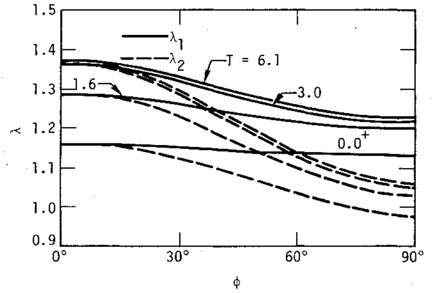

Fig. 3 Principal stretch ratios in an ellipsoidal viscoelastic membrane under constant inflating pressure ($p = 1.0$, $\beta = 0.25$, $\gamma = 0.70$).

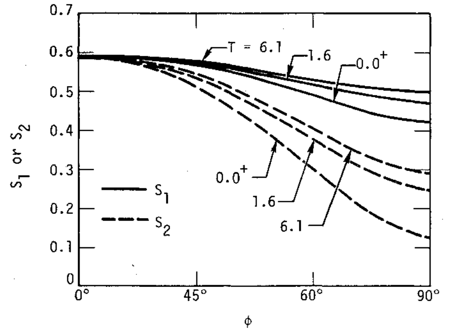

Fig. 4 Principal stress resultants in an ellipsoidal viscoelastic membrane under constant inflating pressure ($p = 1.0$, $\beta = 0.35$, $\gamma = 0.70$).

stant. The variations of the pressure for the relaxation problem are obtained by keeping the deformation unchanged.

Discussion

For elastic neo-Hookean spherical membranes, Green and Adkins[4] stated that an initial rise in pressure will be followed by a fall as inflation proceeds. The same phenomenon is observed in this paper. There are two deformed configurations for a given pressure below the critical pressure. As shown in Fig. 5, when $P = 1.0$, there are two values for λ_0, the principal stretch ratio at the pole: 1.16 and 3.30. The smaller value is for the deformed equilibrium configurations, where the pressure increases as the deformation increases. The larger value is for the deformed equilibrium configuration past the critical point, where the pressure decreases as the deformation increases. This phenomenon is also observed for $t > 0^+$ as shown in Fig. 5. As mentioned in the constitutive equations, the material becomes softer as time elapses for $\gamma < 1$, and the material becomes stiffer for $\gamma > 1$. For the small inflation before the critical point, the deformation increases as the material becomes softer. For large inflations, where the deformation passes the critical point, the deformation decreases as

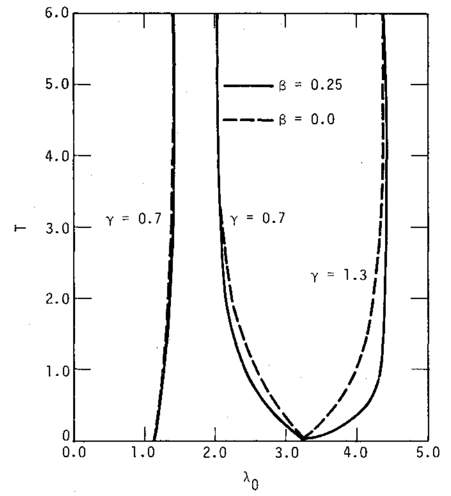

Fig. 5 Deformation history for ellipsoidal viscoelastic membranes under constant inflating pressure.

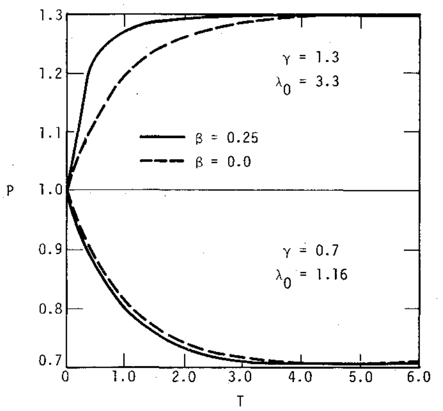

Fig. 6 Inflating pressure history for ellipsoidal viscoelastic membranes under a constant deformation.

the material becomes softer and the deformation increases as the material becomes stiffer.

The constant β governs the effects of deformation and pressure on the response time. The rate change of deformation for the creep problem and the rate change of pressure for the relaxation problem are shown in Figs. 5 and 6.

The rate change of the physical quantities decreases as time increases. In the numerical calculations as exponential time increment is used to save computing time. The equation is

$$T_k - T_{k-1} = m_1 \exp(m_2 k) \tag{26}$$

where m_1 and m_2 are two dimensionless constants. For large deformations,the values assigned to m_1 and m_2 are 0.12 and 0.05, respectively. The solution for each time step takes only a few seconds on the IBM 360.

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